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On the stability of flow in an elliptic pipe which is nearly circular†

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In an earlier paper (Davey 1978) the first author investigated the linear stability of flow in a straight pipe whose cross-section was an ellipse, of small ellipticity e, by regarding the flow as a perturbation of Poiseuille flow in a circular pipe. That paper contained some serious errors which we correct herein. We show analytically that for the most important mode $n = 1$, for which the circular problem has a double eigenvalue c_0 as the 'swirl' can be in either direction, the ellipticity splits the double eigenvalue into two separate eigenvalues $c_0 \pm e^2 c_{12}$, to leading order, when the cross-sectional area of the pipe is kept fixed. The imaginary part of c_{12} is non-zero and so the ellipticity always makes the flow less stable. This specific problem is generic to a much wider class of fluid dynamical problems which are made less stable when the symmetry group of the dynamical system is reduced from S^1 to Z_2.

In the Appendix, P. G. Drazin describes simply the qualitative structure of this problem, and other problems with the same symmetries, without technical detail.

1. Introduction

In this paper we are concerned with the linear stability of laminar flow in a straight pipe whose cross-section is an ellipse. This is an interesting problem because in the extreme case when the ellipticity e decreases towards zero the cross-section becomes more circular and it is generally believed that circular pipe flow is stable to infinitesimal disturbances so that no finite critical Reynolds number exists. In the other extreme case when the ellipticity e approaches the value 1 from below Hocking (1977) has shown that the stability characteristics of the flow are a regular perturbation from those for plane Poiseuille flow which has a critical Reynolds number of about 5772. He found that the critical Reynolds number increased as e decreased below 1 by an amount proportional to $(1-e)^{1/2}$.

Thus, elliptic pipe flow is stable when e is sufficiently small and unstable when e is close to 1; it seems likely therefore that there will exist a critical value of e, namely e_c, such that when $e > e_c$ a critical Reynolds number will exist and when $e < e_c$ then there will be no such critical Reynolds number. In an earlier paper by one of the present authors (Davey 1978; hereinafter referred to as D) an attempt was made to solve the linear stability problem for small values of e by a regular perturbation away from circular pipe flow; however, that paper contained two serious errors.

The first error in D is rather an elementary one: the author kept the length of one of the axes, a, of the ellipse fixed and allowed the other axis of length b to be a function

† With an appendix by P. G. Drazin.

of a and the ellipticity e. However, he used a only as a lengthscale for defining the Reynolds number and the non-dimensional axial wavelength α of the disturbance. This choice of lengthscale is inappropriate because as e varies it does not accommodate the change in the size of the cross-section of the pipe.

The second error in D is more serious although at the same time it is also rather subtle and not easy to detect. The author forgot that, except for the rather special axisymmetric mode, all the eigenvalues of the linear stability problem for circular pipe flow are double eigenvalues because each eigenvalue has associated with it two eigenfunctions, one associated with right-handed 'swirl' in the pipe and one associated with left-handed 'swirl' in the pipe. Now the right-hand side of (19) of D contains terms associated with both 'swirl' directions when the mode number n is such that $n-2 = -n$, i.e. when $n = 1$ which is the most important case of all. This means that the adjoint theory used by D to determine the effect of the ellipticity e at leading order on the complex wave speed c of the disturbance where

$$c = c_0 + e^2 c_1 + O(e^4),$$

(see (16e) in §2), i.e. to determine the coefficient c_1, is erroneous because the author could have used two different adjoint functions which would have led to two different values for c_1. The mathematics in D breaks down completely at this stage.

We shall show that the true situation, for the important case $n = 1$, is that if we allow the lengths of both axes of the ellipse to vary by small amounts ϵ_a, ϵ_b say, so that for example

$$a^2 = a_0^2(1+\epsilon_a), \quad b^2 = a_0^2(1+\epsilon_b),$$

then to leading order in ϵ_a, ϵ_b the expansion for the complex wave speed c takes the form

$$c = c_0 + (c_{11} \pm c_{12})\,\epsilon_a + (c_{11} \mp c_{12})\,\epsilon_b,$$

which may also be written in the form

$$c = c_0 + (\epsilon_a + \epsilon_b)\,c_{11} \pm (\epsilon_a - \epsilon_b)\,c_{12}. \tag{1}$$

In the above two equations $c_1 = c_{11} \pm c_{12}$ and the reason why c_1 can have two values is because there are two adjoint functions and c_1 is the solution of a quadratic equation which has the matrix form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = c_1 \begin{pmatrix} a_+ \\ a_- \end{pmatrix}, \tag{2}$$

where $\{a_+, a_-\}^T$ is the associated eigenvector. (This equation is predicted by the general theory of the Appendix to this paper by P. G. Drazin.)

What we are interested in is the change in the stability characteristics of the flow due to the effect of the ellipticity on the shape of the cross-section of the pipe. We wish to completely eliminate size effects from both our analysis and our numerical calculations. It is clear from (1) that if we choose $\epsilon_a = \epsilon_b$, so that the pipe is still circular but with a different radius, then the c_{12} term will be absent and so the coefficient c_{11} is associated solely with size effects. On the other hand, if we choose $\epsilon_a = -\epsilon_b$ in (1), so that the cross-sectional area of the pipe is preserved (at least to leading order), then the c_{11} term will be absent and so the coefficient c_{12} is clearly associated solely with shape effects.

So, in our analysis which is contained in §2, we carefully choose $\epsilon_a = -\epsilon_b = \frac{1}{2}e^2$ so that the change in the complex wave speed c is due solely to shape effects. The choice for ϵ_a, ϵ_b means that at least to order e^2: (i) the ellipticity of the pipe is e; (ii) the circumference and also the cross-sectional area of the pipe remain the same, and (iii) the volume flux down the pipe remains the same.

Because D chose $\epsilon_a = 0$ and $\epsilon_b = e^2$ his c_1 contained an unwanted size contribution c_{11}, essentially due to the fact that the mean radius of the pipe had increased by a factor $(1+\frac{1}{4}e^2)$ and so therefore had the wavenumber α and the Reynolds number R. Moreover, because he forgot that c is a double eigenvalue, he did not know about c_{12} so it is entirely missing from his paper. So the c_1 which D calculated was due solely to size effects and in fact since $c = c(\alpha, R)$

$$\delta c = \frac{\partial c}{\partial \alpha}\delta\alpha + \frac{\partial c}{\partial R}\delta R,$$

and $\delta\alpha \sim \frac{1}{4}e^2\alpha$ and $\delta R \sim \frac{1}{4}e^2 R$ so it follows that the $c_1 = c_{11}$ which D laboriously calculated is nothing more than

$$\frac{1}{4}\left(\alpha\frac{\partial c}{\partial \alpha} + R\frac{\partial c}{\partial R}\right).$$

The only other fluid dynamical paper which we know of which deals with the perturbation of a double eigenvalue problem together with its attendant double/matrix adjoint theory is by Moore & Saffman (1975) who consider the instability of a straight vortex filament which can support disturbances with both left- and right-helicity; see also Schiff (1968, chap. 8).

Our objective in §2 is to show analytically that the double eigenvalue c_0 for circular pipe flow splits into two separate eigenvalues $c_0 \pm e^2 c_{12}$ for the elliptic problem. In §3 we calculate c_{12} numerically for a wide range of values of α, R and we find that the imaginary part of c_{12} is non-zero. Hence, the effect of the ellipticity must be to make the flow less stable.

2. Linear stability of flow in a pipe with small ellipticity

The steady flow whose stability we wish to examine is that of a viscous incompressible fluid flowing along a straight pipe of elliptic cross-section under the action of a constant pressure gradient. The length of the semi-major axis of the elliptic cross-section is a and the length of the semi-minor axis is b, so that $a \geqslant b$. As mentioned in the Introduction it is convenient to let

$$a^2 = a_0^2(1+\tfrac{1}{2}e^2), \quad b^2 = a_0^2(1-\tfrac{1}{2}e^2), \tag{3}$$

so that for small values of the ellipticity e the square of the ellipticity of the cross-section is e^2 with an error term of order e^4. (In what follows we shall never need to consider any terms of order e^4.)

We suppose that the centreline speed of Poiseuille flow along the pipe is U_0, so that the constant pressure gradient needed to maintain the flow is

$$-2\rho\nu U_0\left(\frac{1}{a^2}+\frac{1}{b^2}\right),$$

where ν is the kinematic viscosity of the fluid and ρ is its density.

We choose a_0, U_0 and a_0/U_0 as the characteristic scales of length, speed and time, respectively, with which we make our quantities non-dimensional. The reference pressure is $\rho\nu U_0/a$. We use non-dimensional Cartesian coordinates (x, y, z) with the x-axis in the direction of the basic flow down the pipe and with the y- and z-axes in the directions of the major and minor axes respectively, so that the boundary of the pipe is given by

$$(1-\tfrac{1}{2}e^2)\,y^2+(1+\tfrac{1}{2}e^2)\,z^2 = 1, \tag{4}$$

and the steady flow down the pipe is

$$U = 1-(1-\tfrac{1}{2}e^2)\,y^2-(1+\tfrac{1}{2}e^2)\,z^2. \tag{5}$$

We shall also use polar coordinates (r,θ) in the cross-sectional plane defined by

$$y = r\cos\theta, \quad z = r\sin\theta, \tag{6}$$

so that (x,r,θ) are cylindrical polars and thus the boundary of the pipe (4) may be written as

$$r^2 = (1-\tfrac{1}{2}e^2\cos 2\theta)^{-1}, \tag{7}$$

and (5) becomes

$$U = 1-r^2-\tfrac{1}{2}e^2r^2\cos 2\theta. \tag{8}$$

We shall consider only linear stability theory and suppose that a disturbance will grow or decay temporally without spatial modulation. Hence it suffices to express the fluid velocity $U_0(u_x,u_r,u_\theta)$ and the pressure $(\rho\nu U_0/a)\mathscr{P}$ in the form

$$\left.\begin{aligned} u_x &= U+\epsilon Eu(r,\theta)+O(\epsilon^2),\\ u_r &= \epsilon Ev(r,\theta)+O(\epsilon^2),\\ u_\theta &= \epsilon Ew(r,\theta)+O(\epsilon^2),\\ \mathscr{P} &= P+\epsilon Ep(r,\theta)+O(\epsilon^2). \end{aligned}\right\} \tag{9}$$

In (9), ϵ is a measure of the amplitude of the disturbance compared with that of the basic flow and

$$E \equiv \exp\{\mathrm{i}\alpha(x-ct)\}, \tag{10}$$

so that the disturbance has wavenumber α in the x-direction, wave speed c_r and temporal growth rate αc_i, where

$$c = c_r+\mathrm{i}c_i. \tag{11}$$

Also, it should be understood that complex conjugates are to be added to the right-hand sides of (9) to cancel the imaginary terms.

If (9) and (10) are substituted into the Navier–Stokes equations and the continuity equation then the terms of order-ϵ yield

$$\left.\begin{aligned} \left(\mathscr{L}+\frac{1}{r^2}\right)u-\mathrm{i}\alpha p &= \tfrac{1}{2}\mathrm{i}\alpha Re^2r^2\cos 2\theta u-2Rr(1-\tfrac{1}{2}e^2\cos 2\theta)\,v-Re^2r\sin 2\theta w,\\ \mathscr{L}v-\frac{\partial p}{\partial r} &= \tfrac{1}{2}\mathrm{i}\alpha Re^2r^2\cos 2\theta v+\frac{2}{r^2}\frac{\partial w}{\partial\theta},\\ \mathscr{L}w-\frac{1}{r}\frac{\partial p}{\partial\theta} &= \tfrac{1}{2}\mathrm{i}\alpha Re^2r^2\cos 2\theta w-\frac{2}{r^2}\frac{\partial v}{\partial\theta}, \end{aligned}\right\} \tag{12}$$

and

$$\frac{\partial v}{\partial r}+\frac{v}{r}+\mathrm{i}\alpha u+\frac{1}{r}\frac{\partial w}{\partial\theta} = 0, \tag{13}$$

where $R = U_0a_0/\nu$ is the Reynolds number and the operator $\mathscr{L}$ is defined by

$$\mathscr{L} \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\alpha^2-\mathrm{i}\alpha R(1-r^2-c). \tag{14}$$

The boundary conditions are

$$\left.\begin{aligned} &u = v = w = 0 \quad \text{when} \quad r = (1-\tfrac{1}{2}e^2\cos 2\theta)^{-1/2}, \\ &\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial \theta} - w = \frac{\partial w}{\partial \theta} + v = \frac{\partial p}{\partial \theta} = 0 \quad \text{when} \quad r = 0. \end{aligned}\right\} \tag{15}$$

We now expand the variables u, v, w and p and also the eigenvalue c in powers of e^2 as follows:

$$u = u_0 + e^2 u_1 + O(e^4), \tag{16a}$$

$$v = v_0 + e^2 v_1 + O(e^4), \tag{16b}$$

$$w = w_0 + e^2 w_1 + O(e^4), \tag{16c}$$

$$p = p_0 + e^2 p_1 + O(e^4), \tag{16d}$$

$$c = c_0 + e^2 c_1 + O(e^4). \tag{16e}$$

If we substitute these expansions into (12)–(15), clearly the leading-order terms u_0, v_0, w_0, p_0 and c_0 will be a linear combination of the eigenmodes of the linear stability problem for circular pipe flow. A typical such eigenmode is of the form

$$\{u_0, v_0, w_0, p_0\} = \{\bar{u}_0(r), \bar{v}_0(r), \bar{w}_0(r), \bar{p}_0(r)\}\exp(\mathrm{i}n\theta), \tag{17}$$

where n is an integer and $\bar{u}_0$, $\bar{v}_0$, $\bar{w}_0$, $\bar{p}_0$ and c_0 are the solutions of the linear ordinary differential Orr–Sommerfeld system

$$\begin{pmatrix} \mathrm{L}+1/r^2 & 2Rr & 0 & -\mathrm{i}\alpha \\ 0 & \mathrm{L} & -2\mathrm{i}n/r^2 & -\mathrm{D} \\ 0 & 2\mathrm{i}n/r^2 & \mathrm{L} & -\mathrm{i}n/r \\ \mathrm{i}\alpha & \mathrm{D}+1/r & \mathrm{i}n/r & 0 \end{pmatrix} \begin{pmatrix} \bar{u}_0 \\ \bar{v}_0 \\ \bar{w}_0 \\ \bar{p}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{18}$$

where $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}r$ and the operator L is defined by

$$\mathrm{L} \equiv \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \left(\frac{1+n^2}{r^2} + \alpha^2\right) - \mathrm{i}\alpha R(1 - r^2 - c_0). \tag{19}$$

The boundary conditions for (18) and (19) are (see Batchelor & Gill 1962)

$$\left.\begin{aligned} &\bar{u}_0 = \bar{v}_0 = \bar{w}_0 = 0 \quad \text{when} \quad r = 1, \\ &\mathrm{i}n\bar{u}_0 = \mathrm{i}n\bar{v}_0 - \bar{w}_0 = \mathrm{i}n\bar{w}_0 + \bar{v}_0 = \mathrm{i}n\bar{p}_0 = 0 \quad \text{when} \quad r = 0; \end{aligned}\right\} \tag{20}$$

also $\bar{u}_0$, $\bar{v}_0$, $\bar{w}_0$ and $\bar{p}_0$ must be finite as $r \to 0$. Equation (18) subject to the boundary conditions (20) can be solved numerically to determine $\bar{u}_0$, $\bar{v}_0$, $\bar{w}_0$, $\bar{p}_0$ and the eigenvalue c_0 for an arbitrary choice of the integer n; see, for example, Lessen, Sadler & Liu (1968) and also Salwen & Grosch (1972). Note that the boundary condition at $r = 0$ takes rather special forms when $n = 0$ or 1, but that otherwise it reduces to

$$\bar{u}_0 = \bar{v}_0 = \bar{w}_0 = \bar{p}_0 = 0.$$

In general the least damped disturbances are those with $n = 1$ (but see Gill 1973) in which case the boundary condition at $r = 0$ becomes $\bar{u}_0 = \bar{v}_0 + \mathrm{i}\bar{w}_0 = \bar{p}_0 = 0$, with $\bar{v}_0$ and $\bar{w}_0$ finite.

Usually, for such a linear stability problem, each eigenvalue will have associated with it a unique eigenfunction, the eigenvalue will be single and the stability of each eigenmode, (17) with a specific value of n, may be considered separately. It is at this stage that D contains the second error: not taking into account that when $n \neq 0$ all the eigenvalues of (18)–(20) are double eigenvalues so that for each eigenvalue c_0 there are two associated eigenfunctions being (17) and also

$$\{u_0, v_0, w_0, p_0\} = \{\bar{u}_0(r), \bar{v}_0(r), -\bar{w}_0(r), \bar{p}_0(r)\} \exp(-in\theta), \tag{21}$$

which is like (17) but with the 'swirl' in the pipe being in the opposite direction: note that (12)–(15) are invariant under the transformation $\theta \to -\theta, w \to -w$.

So at leading order e^0 the eigenfunction associated with the nth mode is a linear combination of (17), (21) so that to evaluate u_1, v_1, w_1, p_1 and c_1 at order e^2 we must take, finally,

$$\begin{aligned}\{u_0, v_0, w_0, p_0\} = {} & a_+\{\bar{u}_0(r), \bar{v}_0(r), \bar{w}_0(r), \bar{p}_0(r)\} \exp(in\theta) \\ & + a_-\{\bar{u}_0(r), \bar{v}_0(r), -\bar{w}_0(r), \bar{p}_0(r)\} \exp(-in\theta). \end{aligned} \tag{22}$$

From (12), (13), (16) and (22) the equations for u_1, v_1, w_1, p_1 and c_1 at order e^2 are

$$\begin{aligned}\left(\mathscr{L}_0 + \frac{1}{r^2}\right) u_1 + 2Rrv_1 - \mathrm{i}\alpha p_1 = {} & a_+\{\mathrm{i}\alpha R(\tfrac{1}{2}r^2 \cos 2\theta - c_1)\, \bar{u}_0 \\ & + Rr\cos 2\theta \bar{v}_0 - Rr \sin 2\theta \bar{w}_0\}\, \mathrm{e}^{\mathrm{i}n\theta} \\ & + a_-\{\mathrm{i}\alpha R(\tfrac{1}{2}r^2 \cos 2\theta - c_1)\, \bar{u}_0 + Rr\cos 2\theta \bar{v}_0 \\ & + Rr \sin 2\theta \bar{w}_0\}\, \mathrm{e}^{-\mathrm{i}n\theta}, \end{aligned} \tag{23a}$$

$$\begin{aligned}\mathscr{L}_0 v_1 - \frac{2}{r^2}\frac{\partial w_1}{\partial \theta} - \frac{\partial p_1}{\partial r} = {} & a_+\{\mathrm{i}\alpha R(\tfrac{1}{2}r^2 \cos 2\theta - c_1)\, \bar{v}_0\}\, \mathrm{e}^{\mathrm{i}n\theta} \\ & + a_-\{\mathrm{i}\alpha R(\tfrac{1}{2}r^2 \cos 2\theta - c_1)\, \bar{v}_0\}\, \mathrm{e}^{-\mathrm{i}n\theta}, \end{aligned} \tag{23b}$$

$$\begin{aligned}\mathscr{L}_0 w_1 + \frac{2}{r^2}\frac{\partial v_1}{\partial \theta} - \frac{1}{r}\frac{\partial p_1}{\partial \theta} = {} & a_+\{\mathrm{i}\alpha R(\tfrac{1}{2}r^2 \cos 2\theta - c_1)\, \bar{w}_0\}\, \mathrm{e}^{\mathrm{i}n\theta} \\ & + a_-\{-\mathrm{i}\alpha R(\tfrac{1}{2}r^2 \cos 2\theta - c_1)\, \bar{w}_0\}\, \mathrm{e}^{-\mathrm{i}n\theta}, \end{aligned} \tag{23c}$$

and

$$\frac{\partial v_1}{\partial r} + \frac{v_1}{r} + \mathrm{i}\alpha u_1 + \frac{1}{r}\frac{\partial w_1}{\partial \theta} = 0, \tag{24}$$

where the operator $\mathscr{L}_0$ is the same as $\mathscr{L}$, see (14), but with c replaced by c_0. Also, from (15), (16) and (22) the boundary conditions for u_1, v_1, w_1 and p_1 are

$$\left.\begin{aligned}(u_1, v_1, w_1) = -\tfrac{1}{4}\cos 2\theta\{a_+\{\bar{u}_0', \bar{v}_0', \bar{w}_0'\}\, \mathrm{e}^{\mathrm{i}n\theta} + a_-(\bar{u}_0', \bar{v}_0', -\bar{w}_0')\, \mathrm{e}^{-\mathrm{i}n\theta}\} \quad & \text{when} \quad r = 1, \\ \frac{\partial u_1}{\partial \theta} = \frac{\partial v_1}{\partial \theta} - w_1 = \frac{\partial w_1}{\partial \theta} + v_1 = \frac{\partial p_1}{\partial \theta} = 0 \quad \text{when} \quad r = 0, & \end{aligned}\right\} \tag{25}$$

where a prime denotes differentiation with respect to r.

The θ-dependence of the forcing terms on the right-hand sides of (23) and (25) may be written as linear combinations of $\exp\{\pm in\theta\}$, $\exp\{\mathrm{i}(\pm n+2)\,\theta\}$ and $\exp\{\mathrm{i}(\pm n-2)\,\theta\}$ so we can seek a solution for u_1, v_1, w_1, p_1 and c_1 which is a linear combination of these six quantities. When we seek those parts of the solution which are proportional to $\exp(in\theta)$ and $\exp(-in\theta)$, the partial differential operator for the homogeneous part of (23) becomes the same as the Orr–Sommerfeld operator in (18) and so has $\{\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{p}_0\}$ and $\{\bar{u}_0, \bar{v}_0, -\bar{w}_0, \bar{p}_0\}$ as eigenfunctions respectively. Hence we can use double adjoint theory to obtain two homogeneous linear equations for a_+ and a_-. For these two equations to have non-trivial solutions the determinant of the coefficient matrix must be zero, see (2), and so c_1 is the solution of a quadratic equation. Thus we obtain the leading-order effect of the ellipticity on the rate of decay of the disturbance.

However, as mentioned in §1, it is only when $n-2$ is as small as $-n$, that is when $n = 1$, that the two simultaneous equations for a_+ and a_- are non-degenerate. When $n \neq 1$ the two equations both imply that $c_1 = 0$ so that the effect of the small ellipticity e on the stability of the flow is only $O(e^4)$ for these cases. Only when $n = 1$ is $c_1 \neq 0$ so that the ellipticity affects the stability at $O(e^2)$. Consequently, hereinafter, we restrict attention to the case $n = 1$.

So, now setting $n = 1$, it follows from (23) and (24) that the vector equation for that part of the solution for u_1, v_1, w_1 and p_1, which is proportional to $\exp\{i\theta\}$ is

$$\begin{pmatrix} \mathrm{L}+1/r^2 & 2Rr & 0 & -\mathrm{i}\alpha \\ 0 & \mathrm{L} & -2\mathrm{i}/r^2 & -\mathrm{D} \\ 0 & 2\mathrm{i}/r^2 & \mathrm{L} & -\mathrm{i}/r \\ \mathrm{i}\alpha & \mathrm{D}+1/r & \mathrm{i}/r & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ p_1 \end{pmatrix} = -\mathrm{i}\alpha R c_1 a_+ \begin{pmatrix} \bar{u}_0 \\ \bar{v}_0 \\ \bar{w}_0 \\ \bar{p}_0 \end{pmatrix} + \tfrac{1}{4}\mathrm{i}\alpha R r^2 a_- \begin{pmatrix} \bar{u}_0 \\ \bar{v}_0 \\ -\bar{w}_0 \\ \bar{p}_0 \end{pmatrix} + \tfrac{1}{2} R r a_- \begin{pmatrix} \bar{v}_0 - \mathrm{i}\bar{w}_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{26}$$

and this equation will only have a solution if the right-hand side is orthogonal to the solution space of the left-hand operator. In order to determine this condition we must find the solution of the associated problem, which is adjoint to the operator of the left-hand side of (26), namely

$$\begin{pmatrix} \mathrm{L}+1/r^2 & 0 & 0 & \mathrm{i}\alpha \\ 2Rr & \mathrm{L} & 2\mathrm{i}/r^2 & -\mathrm{D} \\ 0 & -2\mathrm{i}/r^2 & \mathrm{L} & \mathrm{i}/r \\ -\mathrm{i}\alpha & \mathrm{D}+1/r & -\mathrm{i}/r & 0 \end{pmatrix} \begin{pmatrix} u_a \\ v_a \\ w_a \\ p_a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{27}$$

with boundary conditions

$$\left.\begin{aligned} u_a = v_a = w_a = 0 \quad &\text{when} \quad r = 1, \\ u_a = v_a - \mathrm{i}w_a = p_a = 0 \quad &\text{when} \quad r = 0, \end{aligned}\right\}; \tag{28}$$

also, v_a and w_a must be finite as $r \to 0$.

We solve (27) and (28) numerically and check that the new value which we obtain for c_0 is close to the corresponding value found from the Orr–Sommerfeld formulation (18)–(20). Having found u_a, v_a, w_a and p_a we now left-multiply (26) by

$$\{ru_a, rv_a, rw_a, rp_a\}^T$$

and integrate from $r = 0$ to $r = 1$ to obtain

$$Ba_- = c_1 a_+, \tag{29}$$

where

$$B = \frac{\dfrac{1}{4}\displaystyle\int_0^1 r^3(\bar{u}_0 u_a + \bar{v}_0 v_a - \bar{w}_0 w_a)\,\mathrm{d}r - \dfrac{\mathrm{i}}{2\alpha}\int_0^1 r^2(\bar{v}_0 - \mathrm{i}\bar{w}_0)\, u_a\,\mathrm{d}r + \dfrac{\mathrm{i}}{8\alpha R}[\bar{u}_0' u_a' - \bar{w}_0' w_a']_{r=1}}{\displaystyle\int_0^1 r(\bar{u}_0 u_a + \bar{v}_0 v_a + \bar{w}_0 w_a)\,\mathrm{d}r} \tag{30}$$

Because we have carefully chosen the lengths of the axes of the elliptical cross-section to vary with e to eliminate size effects at order e^2 then both c_{11} and A are zero, see §1.

Now remember that we obtained (29), (30) by focusing our attention on that part of the solution for u_1, v_1, w_1 and p_1 which is proportional to $\exp\{i\theta\}$. In an exactly similar way, by concentrating on that part of the solution which is proportional to $\exp\{-i\theta\}$, and using the other eigenfunction and the other adjoint function, for which the signs of w_0 and w_a are reversed respectively, we obtain

$$Ba_+ = c_1 a_-. \tag{31}$$

Equations (29), (31) yield the two solutions

$$c_1 = B, \quad a_+ = a_- \quad \text{and} \quad c_1 = -B, \quad a_+ = -a_-, \tag{32}$$

so that the temporal growth rate of the disturbance is given by

$$\begin{aligned} \sigma &= \mathrm{Re}\{-\mathrm{i}\alpha(c_0 + e^2 c_1)\} + O(e^4) \\ &= \alpha c_{0i} \pm e^2 \alpha B_i + O(e^4), \end{aligned} \tag{33}$$

where $B = B_r + \mathrm{i}B_i$.

Since numerical calculations indicate that B_i is non-zero it follows immediately from (33) that the effect of the ellipticity is to destabilize the flow; the double eigenvalue when $e = 0$ splitting into two separate eigenvalues for $e > 0$ with one less stable and the other more stable. It is also clear from (32) that in the limit as $e \downarrow 0$ the two fundamental eigenmodes of the elliptical problem are each associated with one of the two axes of the ellipse.

Once c_1 has been found there is no difficulty in principle in successively calculating higher-order terms in (16*a–e*), such as the coefficient c_2 of the term of order-e^4 in (16*e*). In practice, however, this is a formidable task because the number of ordinary differential equations which must be solved numerically increases by an order of magnitude every time an extra term in each of (16*a–e*) is required. (Also, the definitions of the lengths of the axes of the ellipse would need refining so that the ellipticity was still e to $O(e^4)$.)

3. Numerical results

Numerical solutions of the Orr–Sommerfeld equation (18) and the adjoint equation (27) were obtained in exactly the same way as described in D. Runge–Kutta integration was used together with a shooting method and orthonormalization so that calculations could be done for large values of αR.

To integrate (18) and (27), which are both of seventh order, they were first reduced to sixth-order systems by differentiating the fourth row of the matrices and subtracting the result from the second row so that the new second row is a first-order differential equation instead of second order. Initially, near $r = 0$, it is necessary to use a power series expansion before the Runge–Kutta integration is begun. For further details the reader is referred to D.

We have already shown analytically in §2 that the effect of making the pipe elliptical is destabilizing; the numerical results indicate that c_{1i} is not zero. The secondary aim of the numerical results is to obtain an estimate for the critical value of e, i.e. the value above which there will be a critical Reynolds number but below which the flow will be stable. A necessary preliminary to achieving this aim is to determine, given a large fixed value of R as in a laboratory experiment, for which value of α and for which type of

R	α	$-c_{0i}/c_{12i}$
1000	0.75398	0.78139
2500	0.23721	0.61530
5000	0.11909	0.52730
10000	0.05961	0.50245
25000	0.02385	0.49523
50000	0.01193	0.49418
100000	0.00596	0.49392
∞	—	0.4938

TABLE 1. For given R and $n = 1$ we show the smallest possible value of $-c_{0i}/c_{12i} = e^2_{min}$ as defined by (34) and the corresponding value of α; to a very rough first approximation the flow is unstable if $e > e_{min}$.

disturbance (centre, wall or distributed mode) will the effect of the ellipticity be most likely to lead to instability?

To answer this question, for each value of R shown in table 1 we calculated c_0 and c_1 for a very wide range of values of α, and also for the first ten centre modes, wall modes and distributed modes. For each value of R we then found the smallest possible value of $-c_{0i}/c_{1i}$ as α was varied, each kind of mode (centre, wall and distributed) being considered. It is these values of $-c_{0i}/c_{1i}$ which are shown in table 1. (See (32), (33); we choose the c_{1i} which are positive.)

The reason why we are particularly interested in the smallest possible value of $-c_{0i}/c_{1i}$ is as follows: given R, if we ignore the terms of order-e^4 in (33) then a disturbance will be unstable if $c_{0i}+e^2c_{1i} > 0$, i.e. if $e > e_{min}$ where, as a first approximation, we define

$$e^2_{min} \equiv \text{smallest possible value of } -c_{0i}/c_{1i} \text{ for } R \text{ fixed.} \tag{34}$$

Since $c_{0i} < 0$ and $c_{1i} > 0$ then e_{min} will be real.

The first fact which we were able to establish numerically was that the centre modes are only very slightly affected by the ellipticity of the boundary. This is because the centre modes are concentrated near the centre of the pipe so one would not expect them to be affected very much by the slight distortion of a distant boundary. Since the wall modes are concentrated near the boundary one would expect them to be affected much more by the ellipticity than the centre modes. So, although the centre modes are less damped than the wall modes for circular pipe flow, all the values of $-c_{0i}/c_{1i}$ found for the centre modes are so very large that we shall not discuss these modes any further.

Next, keeping R fixed and setting $\alpha = 1$, we calculated $-c_{0i}/c_{1i}$ for the ten least-stable wall modes, we then allowed α to vary over a very wide range for each of the ten modes. We found that $-c_{0i}/c_{1i}$ was always a minimum for the distributed mode associated with one of the least-damped wall modes, the key value of α being very small. (A wall mode gradually changes into a distributed mode when α becomes sufficiently small.) Table 1 contains the principal results which we obtained (for the case $n = 1$ of course). The columns headed $-c_{0i}/c_{1i}$ are the smallest possible values as α varies, with R fixed, so these columns could also have been headed e^2_{min} as defined by (34); the column which is headed α are the corresponding wavenumbers.

We see from table 1 that e_{min} is a monotonically decreasing function of R with a finite limit as $R\to\infty$; this limiting value was found from the convergence of algebraic and exponential Shanks transforms. This means that, given R, the flow will be unstable if $e > e_{min}$. Alternatively it means that given $e = e_{min}$, R will be the critical Reynolds number and α will be the critical wavenumber at the 'nose' of the neutral-stability curve. Note that when R is large α is also a monotonically decreasing function of R, in fact $\alpha \sim R^{-1}$ and the critical wavelengths are much larger than the mean radius of the pipe.

It is clear from table 1 that we are concerned with values of e^2 of about 0.5 and we can only really *just* call this small, and so the justification for neglecting terms of order-e^4 in (33) is at best plausible. Hence, we must remember that (34) only gives us a very rough first approximation to the relation which must hold between the Reynolds number R and the ellipticity e for a neutral disturbance to exist. What we can say with certainty is that since $c_1 = \pm B, B_i \neq 0$, the effect of the ellipticity is to make the flow less stable.

In general we found that the contributions to c_{1i} from the two integrals in the numerator of (30) were negative and of the same order of magnitude, so that the last term in the numerator dominates to make the value of c_{1i} positive. This last term also dominates the value of c_{1r}, which is negative, so that one effect of the ellipticity on the destabilizing solution is to decrease the phase speed. For example, if $R = 100000$ and $\alpha = 0.00596$, then $c_0 = 0.5720-0.3311\mathrm{i}$, $c_1 = -0.3089+0.6703\mathrm{i}$ and the three contributions to c_1 from (30) are, in order, $1.6200-0.8174\mathrm{i}$, $1.4014-0.4763\mathrm{i}$ and $-3.3302+1.9639\mathrm{i}$.

4. Concluding remarks

We have examined the linear stability of flow in a straight pipe whose cross-section is an ellipse of small ellipticity e by considering the flow to be a perturbation away from flow in a circular pipe of the same size. We have shown analytically that to leading order the effect of the ellipticity is to make the flow less stable and that, given e and a fixed Reynolds number R, as in a laboratory experiment, the mode which is most likely to lead to instability is the distributed mode associated with one of the least-damped wall modes for the case $n = 1$; the key value of α which maximizes the effect of the ellipticity being of order R^{-1} when R is large.

Moreover, we have obtained a very rough estimate which plausibly suggests that flow in an elliptic pipe will be unstable and a critical Reynolds number will exist if e^2 is larger than about one-half, that is, if the aspect ratio a/b of the pipe is larger than about 1.5. This value may be compared with the value of 3.2 found by Tatsumi & Yoshimura (1990) for the critical aspect ratio for the stability of flow in a rectangular duct. It is interesting to note from their paper that contours of constant axial velocity for the basic laminar flow are almost ellipses.

There are two other interesting points worthy of note. First, the fact that c_{12} is non-zero only for $n = 1$ implies that the effect of the ellipticity only occurs at order-e^4 for other values of n. In a way this result supports the long-held conviction that $n = 1$ is also the most important mode as regards the stability of circular pipe flow. Secondly, in the limit as $e \downarrow 0$ the fundamental eigenmodes for the elliptical problem corresponding to $\{a_+, a_-\} = (1, 1)$ and $(1, -1)$ have associated velocity fields proportional to

$$\{\bar{u}_0(r)\cos\theta, \bar{v}_0(r)\cos\theta, \mathrm{i}\bar{w}_0(r)\sin\theta\} + \text{c.c.},$$

and

$$\{\bar{u}_0(r)\sin\theta, \bar{v}_0(r)\sin\theta, -\mathrm{i}\bar{w}_0(r)\cos\theta\} + \text{c.c.},$$

respectively; so the two eigenmodes are each associated with different axes of the ellipse.

A numerical study of the linear stability of the elliptical pipe flow problem for finite ellipticity is currently in progress by Dr R. R. Kerswell and one of the authors (A.D.); preliminary calculations have confirmed the small-ellipticity results presented in §3 of this paper.

The authors wish to express their deep gratitude to Professor P. G. Drazin for his continual interest and encouragement throughout the preparation of this paper. We also thank the referees for their many helpful comments.

Appendix. The structure of the stability problem

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The main paper presents quantitative results for a specific problem which is one of a wide class of problems of hydrodynamic stability sharing the same symmetries. So it is useful to describe below the qualitative structure of the results in a general way with no technical detail. This illuminates the paper and points the way to solve other problems.

The dimensionless problem of the development of all initial perturbations of a basic flow in an elliptic pipe is, for given values of the Reynolds number R and eccentricity e, not only invariant under the continuous group of all dimensional scalings (summarized in §1), but also is invariant under the continuous group of all translations in the axial direction and under the group Z_2 of reversals of the axial coordinate x. Again, it is invariant under translations of time (but is only invariant under reversal of time if the fluid is inviscid). A particular perturbation may or may not possess these symmetries.

For a circular pipe ($e = 0$) the problem is axisymmetric, i.e. invariant under all transformations belonging to the continuous group S^1 of rotations about the axis and the group Z_2 of reversals of the azimuthal angle θ. However, for an elliptic pipe ($e \neq 0$) the problem is symmetric only under reflections $Z_2 \times Z_2$ in the major and minor axes of the ellipse, and under interchange of the major and minor axes.

The development of perturbations of the basic pipe flow is modelled as a set of partial differential equations and boundary conditions, which is plausibly equivalent to an infinite set of nonlinear ordinary differential equations, say

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{a}, R, e).$$

For example, a perturbation of the velocity might be represented as a spectral expansion in the set of normal modes $\mathrm{e}^{\mathrm{i}(\alpha x+n\theta)}\boldsymbol{u}_m(r,\alpha,n,R,e)$ of the *linearized* problem with complex amplitudes $a_{\alpha mn}$ for real $\alpha, m = 1,2,\ldots$ and $n = 0, \pm 1, \pm 2, \ldots$, where $\boldsymbol{u}_m$ is the mth radial eigenfunction, such that $\boldsymbol{a}$ has components $a_{\alpha mn}$ to specify the perturbation. The invariance of the problem under axial translations then implies that

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \boldsymbol{g}(|\boldsymbol{a}|^2, R, e)\,\boldsymbol{a},$$

for some matrix $\boldsymbol{g}$ depending on $\boldsymbol{a}$ only through the squares of the moduli $|a_{\alpha mn}|^2$ of the components of $\boldsymbol{a}$.

Now the dimensionless form of equation (7) of the pipe wall is

$$r = 1 + e^2 F(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta}, e^2),$$

and of equation (8) of the basic flow is

$$U = 1 - r^2 + e^2 r^2 G(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta}),$$

for certain functions F, G. As in the main paper take $m = 1$ (because it is plausible that all modes with $m \geqslant 2$ are very stable). It is convenient to omit explicit mention of α, m, R. Then

$$\frac{\mathrm{d}a_n}{\mathrm{d}t} = \sum_{p=-\infty}^{\infty} g_{np}(|\boldsymbol{a}|^2, e^2)\, a_p \tag{A 1}$$

for $n = 0, \pm 1, \pm 2, \ldots$. Also, it is well known that the linearized problem for a circular pipe gives the form

$$\frac{\mathrm{d}a_n}{\mathrm{d}t} = \sigma_n a_n + O(|\boldsymbol{a}|^2 \boldsymbol{a}, e^2 \boldsymbol{a}) \tag{A 2}$$

as $e, |\boldsymbol{a}| \rightarrow 0$, where $\sigma_{-n} = \sigma_n$ by symmetry in $\pm\theta$. Therefore

$$\frac{\mathrm{d}a_n}{\mathrm{d}t} = \sigma_n a_n + e^2(A_n a_n + B_{n,\,n+2}\, a_{n+2} + B_{n,\,n-2}\, a_{n-2}) + O(|\boldsymbol{a}|^2 \boldsymbol{a}, e^4 \boldsymbol{a}) \tag{A 3}$$

for some complex interaction coefficients $A_n, B_{n,p}$ (dependent on α, R) because F, G depend on $\mathrm{e}^{\pm 2\mathrm{i}\theta}$ only. By symmetry, $A_{-n} = A_n, B_{-n,-p} = B_{n,p}$.

The modes with $n = \pm 1$ are likely to be the most unstable for smallish values of e because they are known to be the least stable when $e = 0$. So it is plausible to consider only the interaction of these two modes, for which (A 3) reduce to

$$\left.\begin{aligned} \frac{\mathrm{d}a_1}{\mathrm{d}t} &= \sigma_1 a_1 + e^2(Aa_1 + Ba_{-1}), \\ \frac{\mathrm{d}a_{-1}}{\mathrm{d}t} &= \sigma_1 a_{-1} + e^2(Ba_1 + Aa_{-1}), \end{aligned}\right\} \tag{A 4}$$

approximately, where $\sigma_1 = -\mathrm{i}\alpha c_0$, say, $A_{-1} = A_1, = A$, say, and $B_{1,-1} = B_{-1,1}, = B$, say, On taking normal modes with $a_+ = a_1, a_- = a_{-1} \propto \mathrm{e}^{-\mathrm{i}\alpha ct}$ and $c = c_0 + e^2 c_1 + O(e^4)$ as $e \rightarrow 0$, we have equation (2) in the notation of the main paper.

These simple ideas show how to anticipate the structure of the detailed calculations when $e, \boldsymbol{a}$ are not small, and also the structure of other stability problems with the same symmetries. They were originally adumbrated in a referee's report of this paper, independently, after the submission but before the publication, of similar ideas by Guckenheimer & Mahalov (1992) on the *weakly nonlinear* stability of a *Hamiltonian* system. In this paper the system is strongly dissipative, but the other symmetries are the same as those assumed by Guckenheimer & Mahalov, so their paper is also helpful in understanding the structure of the weakly nonlinear stability of flow in an elliptic pipe.

Note added in Proof. It has been pointed out to us by Professor J. T. Stuart that equations (A4) of the Appendix are related to the equations which govern the Benjamin–Feir instability and the Eckhaus instability, both of which constitute examples of side-band instabilities.

The relationship arises as follows: in the Benjamin–Feir and Eckhaus side-band mechanisms, the harmonic, $\exp(2\mathrm{i}kx)$ plays a central *nonlinear* role in causing an interaction between two side-band waves neighbouring $\exp(\mathrm{i}kx)$. In the present *linear* problem, however, the role of that harmonic is taken on by the $\cos 2\theta$ term in the boundary description (8); it is this term which causes the interaction between the two basic modes (17) and (21) of the present paper (with $n = 1$).

For details the reader is referred to Stuart & DiPrima (1978), equations (2.8) and (3.13), to which equations (A4) are related.

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